Driven interfaces in quenched disorder at critical depinning

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We reexamine the problem of interfaces driven by a bulk force in quenched disorder. We find that the Bruinsma-Aeppli equation is not compatible with the physical Hamiltonian by having one extra symmetry. We discuss the definition of the critical depinning and show that at criticality, equations with and without the nonlinear term have very different scalings. A stretched exponential law is found for the derivative distribution when lateral growth is absent.

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Interfaces in random media driven by bulk forces have attracted much interest in various domains in the last years. Bruinsma and Aeppli [1] first started the subject by introducing a partial differential equation in their study of the two-dimensional (2D) random field domain wall problem. Koplik and Levine [2] applied the same equation to a much larger scale problem: the driven water-oil interface in porous rocks. The problem has recently been revived in the connection of interface growth phenomena [3], especially in experimental realizations of interface growth [4–6].

Significant progress has been achieved in understanding the critical scaling behavior of prototype lattice models of driven interfaces in quenched disorder. The stationary interface configurations were shown to be related to directed percolation clusters [7,8]. Later, Olami, Procaccia, and Zeitak [9] and Tang and Leschhorn [10] were able to derive almost all other exponents from χ alone, using a scaling theory. In this article we shall compare two interface equations: the first is the Edwards-Wilkinson (EW) equation in quenched disorder, the second is the same but with an extra nonlinear term.

Let us start by considering the Bruinsma-Aeppli Hamiltonian in $d=1+1$ dimensions:

$$H[h(x),f]=\int dx\frac{\nu}{2}|\nabla h(x)|^2 + \int dx\int_0^{h(x)}[\eta(h',x)-f]dh' , \tag{1}$$

where $h(x)$ is the domain wall's position. The first term on the right hand side is due to constant surface tension ν, the second term represents the contribution from random pinning in the bulk below the domain wall, and f is a constant representing the external driving force. If we ignore thermal noise, traditionally [1] we would have the following dynamic equation for $h(x,t)$:

$$\dot{h}(x,t)=-\frac{\delta H[h,f]}{\delta h} , \tag{2}$$

which leads to the well-known Bruinsma-Aeppli-Koplik-Levine (BAKL) equation

$$\dot{h}(x,t)=\nu\nabla^2 h(x,t)-\eta(h,x)+f , \tag{3}$$

which is at the basis of much of the current research.

We want to point our here that Eq. (3) is inadequate to describe interfaces driven by an external bulk force. To see this, note that Eq. (3) enjoys the symmetry

$$h'(x)=h(x)+Cx , \tag{4}$$

C being a constant, where for independently distributed disorder statistical equivalence $\eta(h',x)\Longleftrightarrow\eta(h,x)$ is assumed. However, this symmetry is not allowed by the term proportional to f in Eq. (1), as can easily be verified. How can the unsolicited symmetry be smuggled into our final equation? We shall see that it is Eq. (2) which is at fault.

It has been known for some time [11,12] that the quantities on both sides of Eq. (2) do not point in the same direction: in the h-x plane, $\dot{h}$ is in the vertical direction (that of h), while $-\delta H[h]/\delta h$ is in the local normal direction to the interface. To establish the correct relationship between the two, one should consider either equating the vertical projections

$$\dot{h}=-\sqrt{1+|\nabla h|^2}\frac{\delta H[h,f]}{\delta h} , \tag{5}$$

or, which is totally equivalent, equating their projections in the normal direction. This was first observed in the kinetic interface context [11,12] but should also be valid for the quenched disorder case. Like the Hamiltonian (1), Eq. (5) does not obey the symmetry transformation (4). To leading order we therefore obtain

$$\dot{h}(x,t)=\nu\nabla^2 h+\lambda|\nabla h|^\alpha-\eta(h,x)+f , \tag{6}$$

where $\alpha=2$, and we have deliberately introduced a different constant λ and left undecided its physical value.

First of all one should learn if the two equations give equivalent scaling behaviors. Should the answer be in the affirmative, then we should prefer the simpler equation (3) over the more complete one (6). Indeed, past work [13,14] shows that in the strong driving regime, i.e., in the unpinned phase, both scale the same way, beyond crossover lengths, and the exponent is the traditional one [15]. This is because that nonlinear term is dynamically generated. Near and at the critical depinning transition, the situation is less clear. One early work [16] suggests that both scale in the same way at criticality. Recent nu-

merical work with various interface models shows, however, that the presence of the nonlinear term makes a big difference in scaling behavior: Eq. (3) results in a super-rough surface ($\chi > 1$), while Eq. (6) yields the well defined self-affine exponent $\chi \simeq 0.63$, and can be well explained by relating to the directed percolation problem [17,18].

Besides the two models discussed above, there are also two different approaches to the critical depinning transition. The traditional one is to carefully choose a suitable driving force f such that the interface just overcomes quenched random pinning forces. This value of f is called the critical force f_c. A recent approach inspired by Sneppen [19] by construction only moves a single site facing the least resistance. We shall argue that, strictly speaking, in finite systems no such a constant f_c exists and any presumed critical value of the driving force necessarily is above criticality. Below we define criticality as the slowest possible interface motion by suitably adjusting the instantaneous driving force f, and we shall see that in real experiments this definition of criticality is natural and realizable.

Let us reconsider one typical interface displacement experiment [3,4]. Pressured water is injected into a porous medium, and the water-air interface advances. Imagine that we want to approach the critical depinning by pumping water so slowly that at most one site on the interface is moving forward. Each time when there is a breakthrough there is a sudden volume gain, the pressure field experiences a temporary drop, and the interface stops advancing. However, water is being still slowly pumped in and the pressure field builds up again until another site with least resistance breaks through and this process repeats itself. It is well known that experimentally it is easier to maintain the constant flow pumping rate than to maintain the constant pressure, therefore the above definition happens to be the most experimentally relevant. This scenario is the slowest one, and thus properly defines criticality, very much in the spirit of self-organized criticality (SOC) [20]. The critical driving force is uniform in space but a function of time, as illustrated in Fig. 1.

The instantaneous critical driving force fluctuates around a constant value f_0 if averaged over a long time. Needless to say, the amplitude of the fluctuations is extremely small for large systems. In fact it can be readily estimated as

$$\Delta f = |f_c(t) - f_0| \sim L^{-1/\xi_\parallel} , \tag{7}$$

where $\xi_\parallel$ is one of the directed percolation exponents, and L is the system size in the transversal direction. f_0 is the closest equivalent to the traditional critical depinning force f_c, however, it cannot be used to drive an interface since there is always a time in which the interface requires a stronger push to overcome pinning. For that reason any cleverly chosen constant f would not do the job. In order to be in the depinning phase, a constant driving force necessarily has to lie above the fluctuation region. However, such a value is not really critical, since for much of the time driving is over criticality by an amount of about Δf.

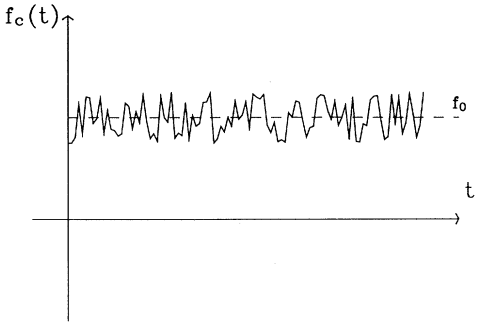

FIG. 1. Critical driving force $f_c(t)$ vs t (solid line). The dashed line represents the average value f_0 of $f_c(t)$ over long times. The critical driving force reduces to f_0 in the limit of an infinite large system.

Having defined the model and the critical depinning, let us investigate whether the above two equations share the same scaling behavior at criticality. For Eq. (6) with $\alpha = 2$, we have clearly confirmed the familiar roughening exponent $\chi \simeq 0.63$, agreeing with previous results on similar models [7–10]. Olami, Procaccia, and Zeitak [17] first considered such a nonlinear equation analytically as well as numerically. They derived the above χ using a directed percolation related scaling theory. In our own simulation, we found that the directed percolation universality class is very large. As a matter of fact, we found that any nonlinear term ($\alpha \geq 1$) which violates the symmetry (4) will make the model scale in the same universality class.

Recently it has been reported by Roux and Hansen [21] that Eq. (3) scales super-roughly, i.e., $\chi \simeq 1.20$; this is also reported by Olami, Procaccia, and Zeitak [17]. We confirm their results and observe that the interface configurations are actually well defined self-affine fractals with the intrinsical exponent $\chi_1 \simeq 0.85$ for a fixed system size L, as also reported in [21]. Traditionally it is assumed that both the height-height correlation function

$$G(r) = \sqrt{\langle [h(x,t) - h(x+r,t)]^2 \rangle} \tag{8}$$

and

$$W(L) = \left\langle \frac{1}{L} \sum_{x=1}^{L} [h(x,t) - \langle h(x,t) \rangle]^2 \right\rangle^{1/2} \tag{9}$$

scale the same way, i.e.,

$$G(r) \sim r^{\chi_1}, \quad W(L) \sim L^{\chi_2} , \tag{10}$$

where $\chi_1 = \chi_2$ and $\langle\ \rangle$ denotes the sample average and $0 \leq r \leq L$. For strictly self-affine interfaces with $\chi_2 < 1$, this can be rigorously verified. Indeed for the nonlinear equation (6) both correlations yield the same roughening exponent $\chi_1 = \chi_2 \simeq 0.63$, as we have verified numerically. The width $W(L)$ dependence on the system size L can have super-rough scaling ($\chi_2 > 1$), this is already known in studying the following equations by Wolf and Villain [22] and Das Sarma and Tamborenea [23], among others:

$$\dot{h}(x,t) = -\nu \nabla^4 h(x,t) + \eta(x,t) . \tag{11}$$

Super-rough interfaces have diverging derivatives. Therefore the width $W(L)$ does not represent the self-

affine scaling nature since the basic step is a diverging quantity [21]

$$|h(x+1,t)-h(x,t)| \sim L^{\beta}, \quad \beta > 0 .$$

That super-rough scaling with two roughening exponents is not contradictory [24], since the height-height correlation function should be expressed as

$$G(r) \sim r^{\chi_1} L^{\chi_2 - \chi_1} . \tag{12}$$

If the correlation length r coincides with the system size L, we recover $G(r=L)=W(L)$. In Fig. 2 we report the scaling results for both the self-affine roughening exponents $\chi_1 \simeq 0.85$ and $\chi_2 \simeq 1.20$, in agreement with Roux and Hansen [21].

We set out to study the nature of such diverging derivatives. In Fig. 3 we plot the histogram of $\Delta h = |h(x+1,t)-h(x,t)|$ for the interfaces in the stationary phase. We find that the distribution $p(\Delta h)$ does not follow a power law, as in the Lévy flight case [25]; rather it follows a stretched exponential law:

$$p(\Delta h) \sim \exp\{-A(\Delta h)^{\gamma}\} , \tag{13}$$

where A is a constant and γ is estimated to be $\gamma \simeq 1.75 \pm 0.06$. This type of stretched exponential law has recently attracted considerable attention [26]; Krug [27] recently found similar laws for other super-rough surfaces.

Finally, we want to point out an instability inherent in

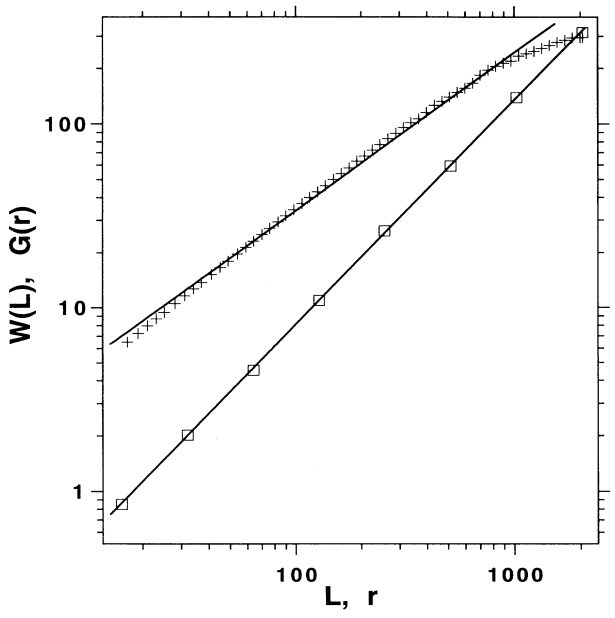

FIG. 2. Scaling of the correlation functions $W(L)$ and $G(r)$ of the fluctuating interface for the EW equation with quenched noise in the saturation phase. We find two different scalings. (a) Plot of $W(L)$ vs L (□). L is the lattice size ($d=1+1$). (b) Plot of $G(r)$ vs r (+). r is the width of a window in a lattice of amplitude L, ($r \leq L$). The super-rough interface growth is evident from the scaling of $W(L)$: $\chi_2 \simeq 1.20 \pm 0.01$. The scaling of $G(r)$ shows the self-affine nature of the interface: we find $\chi_1 \simeq 0.85 \pm 0.02$.

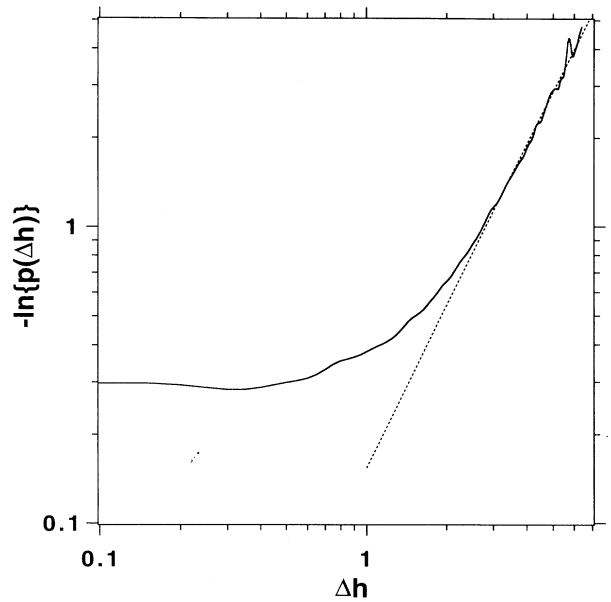

FIG. 3. Histogram of the distribution of Δh for the interface at the stationary phase. The function $-\ln(p(\Delta h))$ vs Δh is shown, on a double-log scale. The numerical estimated value of the stretched exponent γ (see text) is $\gamma \simeq 1.75 \pm 0.06$.

the nonlinear equation (6), which makes the equation as it stands ill defined. Suppose that at a given moment a rare event makes the interface at one site rougher than elsewhere. Then the nonlinear term suddenly becomes dominant over the surface tension term. As a consequence, the minimal resistance will always happen at this site, making the interface infinitely rough.

Let us consider the range where $\lambda \neq 0$ and $\nu \neq 0$. To choose the minimal resistance we have to examine

$$f(x,t) = \eta(x,h) - \nu \nabla^2 h(x,t) - \lambda |\nabla h(x,t)|^{\alpha} . \tag{14}$$

To illustrate this hidden instability, let us consider the simplest case in $d=1+1$ dimensions: $h(x,t)=0$ on all the sites except x_0, where $h(x_0,t)=\xi>0$. All the following considerations can be easily generalized to a d-dimensional system as well.

It is straightforward to show that the disorder averaged value [$\langle \eta(h,x) \rangle = 0$]

$$\langle f(x_0,t) \rangle = 2\nu\xi - \lambda \xi^{\alpha} \tag{15}$$

represents the bias on the chance that the site x_0 will be chosen to have minimal resistance. We have to compare this bias to that of its neighbors:

$$\langle f(x_0 \pm 1,t) \rangle = -\nu\xi - \frac{\lambda}{2}\xi^{\alpha} , \tag{16}$$

while it is zero in other places. Therefore the relative bias for the site x_0 is

$$\Phi(\xi) = \langle f(x_0,t) - f(x_0 \pm 1) \rangle = \left[3\nu - \frac{\lambda}{2}\xi^{\alpha-1} \right] \xi . \tag{17}$$

We see that for small values of ξ the bias $\Phi(\xi)$ is positive, thus the site at x_0 has less of a chance to move forward

than the sites at $x_0 \pm 1$. Hence the interface is stable. On the other hand, for ξ larger than a critical value

$$\xi^* = \frac{6\nu}{\lambda} \quad (\alpha = 2) ,$$

the right-hand side of (18) becomes negative and the chance to advance for the site x_0 is enhanced, i.e., it is favorable to have $\xi \to \xi + 1$. For $\alpha > 1$ this is devastating since ξ keeps growing. We depict the bias function $\Phi(\xi)$ in Fig. 4 for $\alpha = 2$. The region of ξ where $\Phi(\xi)$ is positive is declared safe. In the $\Phi(\xi)$ negative region the uncontrolled growth of ξ at the site x_0 prevails.

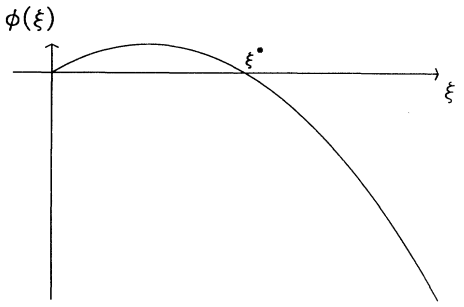

FIG. 4. Shape of the bias function $\Phi(\xi)$ vs ξ in the simplest case $\alpha = 2$ and $d = 1 + 1$. The critical value ξ^* separates stable and unstable regions.

Conclusion: in this article we have reexamined the problem of driven interfaces in quenched disorder. We found that the traditional interface equation (EW) is incompatible with the physical Hamiltonian. A correct treatment necessitates the introduction of a nonlinear term for the interface equation. We also have introduced a precise definition of the critical depinning transition and argued that a fluctuating driving force is experimentally realistic and theoretically appropriate. Interfaces scale very differently with and without the nonlinear term, and the latter is shown to be super-rough. A stretched exponential law is found for the distribution of the jumps Δh.

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